2010 INP

Student name / number ....

Marks

Question 2 **Begin a new booklet**

(a) If $z = 3 - i$ and $w = 1 + 2i$, find in the form $a + ib$, where a and b are real, the values of

(i) $z - 2w$. **1**

(ii) $z\overline{w}$. **1**

(iii) $\dfrac{z}{w}$. **1**

(b)(i) Show that $\tan\frac{\pi}{12} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$. **1**

(ii) Express $z = \left(\sqrt{3}+1\right) + \left(\sqrt{3}-1\right)i$ in modulus argument form. **2**

(iii) Express z^6 in the form $a + ib$, where a and b are real. **1**

(c)(i) On an Argand diagram, shade the region where both $|z - 1 - i| \le \sqrt{2}$ and $0 \le \arg z \le \frac{\pi}{4}$. **2**

(ii) Find in simplest exact form the area of the shaded region. **2**

(d)(i) If $z_1 = r(\cos\alpha + i\sin\alpha)$ and $z_2 = r(\cos\beta + i\sin\beta)$ show that $\left|\sqrt{z_1 z_2}\right| = r$ and $\arg\left(\sqrt{z_1 z_2}\right) = \frac{1}{2}(\alpha + \beta) + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$. **2**

(ii) If $0 < \alpha < \beta < \frac{\pi}{2}$, show on an Argand diagram the points A, B, C, D and E such that vectors $\overrightarrow{OA}, \overrightarrow{OB}, \overrightarrow{OC}$ represent z_1, z_2, $z_1 + z_2$ respectively, and $\overrightarrow{OD}, \overrightarrow{OE}$ represent the two square roots of $z_1 z_2$. **2**

Question 2
a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • writes value of $z-2w$	1
ii • writes value of $z\overline{w}$	1
iii • writes value of $\frac{z}{w}$	1

Answer

$z=3-i$ and $w=1+2i$

i. $z-2w=(3-i)-2(1+2i)=1-5i$

ii. $z\overline{w}=(3-i)(1-2i)=1-7i$

iii. $\frac{z}{w}=\frac{(3-i)(1-2i)}{(1+2i)(1-2i)}=\frac{1-7i}{1^2+2^2}=\frac{1}{5}-\frac{7}{5}i$

b. Outcomes assessed : H5, E3

Marking Guidelines

Criteria	Marks
i • applies result for tan of a difference	1
ii • finds the modulus of z	1
• deduces z has argument $\frac{\pi}{12}$ and writes z in modulus argument form	1
iii • finds the 6^{th} power in required form	1

Answer

i. $\tan\frac{\pi}{12}=\tan(\frac{\pi}{3}-\frac{\pi}{4})=\frac{\tan\frac{\pi}{3}-\tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}}=\frac{\sqrt{3}-1}{1+\sqrt{3}.1}=\frac{\sqrt{3}-1}{\sqrt{3}+1}$

ii. $z=(\sqrt{3}+1)+(\sqrt{3}-1)i \Rightarrow |z|=\sqrt{8}=2\sqrt{2}$ and $\arg z=\alpha$ where $\tan\alpha=\frac{\sqrt{3}-1}{\sqrt{3}+1}$, $0<\alpha<\frac{\pi}{2}$.

$\therefore\ z=2\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$

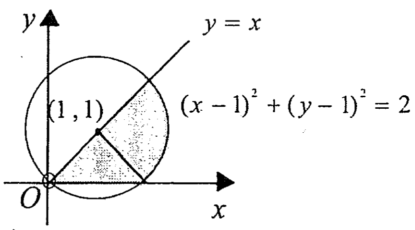
iii. $z^6=2^9\left(\cos\frac{6\pi}{12}+i\sin\frac{6\pi}{12}\right)=512i$

c. Outcomes assessed : P4, E3

Marking Guidelines

Criteria	Marks
i • realises region lies inside circle, centre (1, 1) and radius $\sqrt{2}$	1
• sketches region bounded by circle, x-axis and line $y=x$, excluding origin O	1
ii • realises region comprises a right triangle and a quarter circle, finding the area of one part	1
• adds second part to give exact area	1

Answer i.



ii.

Area is $\frac{1}{2}\sqrt{2}.\sqrt{2}+\frac{1}{4}\pi\left(\sqrt{2}\right)^2=1+\frac{\pi}{2}$

2d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • uses de Moivre's theorem to write down the product z_1z_2 and deduce that $\left\|\sqrt{z_1z_2}\right\| = r$	1
• deduces the possible values of $\arg(z_1z_2)$	1
ii • shows A, B, C on an Argand diagram, with $OACB$ forming a rhombus.	1
• shows D, E collinear with O and C, and $OA = OB = OD = OE = r$	1

Answer

i. $z_1 = r(\cos\alpha + i\sin\alpha)$, $\quad z_2 = r(\cos\beta + i\sin\beta)$

Using de Moivre's theorem,

$$z_1z_2 = r^2(\cos(\alpha+\beta) + i\sin(\alpha+\beta))$$

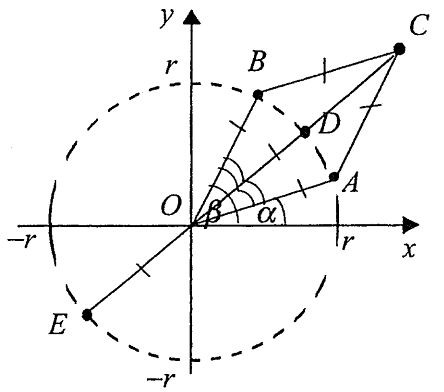
$$= r^2\left\{\cos\left(\frac{\alpha+\beta+2n\pi}{2}\right) + i\sin\left(\frac{\alpha+\beta+2n\pi}{2}\right)\right\}^2$$

where $n = 0, \pm 1, \pm 2, \ldots$

Now $|z_1z_2| = r^2$, hence $\left|\sqrt{z_1z_2}\right| = r$. Also

$\arg\left(\sqrt{z_1z_2}\right) = \frac{1}{2}(\alpha+\beta) + n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

ii. If $0 < \alpha < \beta < \frac{\pi}{2}$, the two square roots of z_1z_2 have principal arguments $\frac{1}{2}(\alpha+\beta)$, $\frac{1}{2}(\alpha+\beta) - \pi$.



Question 3

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• solves $P'(x) = 0$ to find the possible double zeros of $P(x)$	1
• finds c if 3 is a double zero of $P(x)$	1
• finds c if 1 is a double zero of $P(x)$	1

Answer

$P(x) = x^3 - 6x^2 + 9x + c$

$P'(x) = 3x^2 - 12x + 9$

$\quad = 3(x-3)(x-1)$

$\therefore P'(x) = 0$ for $x = 3$ or $x = 1$

$\therefore P'(3) = P(3) = 0 \Leftrightarrow 27 - 54 + 27 + c = 0 \Leftrightarrow c = 0$

and $P'(1) = P(1) = 0 \Leftrightarrow 1 - 6 + 9 + c = 0 \Leftrightarrow c = -4$

$\therefore P(x)$ has a double zero if and only if $c = 0$ or $c = -4$.