QUESTION TWO (15 marks) Use a separate writing booklet. **Marks**

(a) Let $z = 3 - 4i$ and $w = 2 + i$. Find, in the form $x + iy$:

(i) $z + iw$ $\boxed{1}$

(ii) $z\overline{w}$ $\boxed{1}$

(b) Let $\alpha = 1 - i$.

(i) Write α in modulus-argument form. $\boxed{1}$

(ii) Hence show that $\alpha^4 + 4 = 0$. $\boxed{2}$

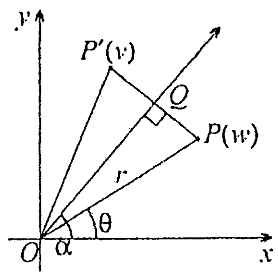
(c) Let $z = x + iy$ and $w = 1 - \dfrac{2i}{z}$.

(i) Write w in the form $a + ib$. $\boxed{2}$

(ii) For what value of z is w undefined? $\boxed{1}$

(iii) Given that w is purely imaginary, describe the locus of z. $\boxed{2}$

(d)



In the Argand diagram above, P represents the complex number $w = r\,\mathrm{cis}\,\theta$. Q is that point on the ray $\arg(z) = \alpha$ such that $\angle PQO = \frac{\pi}{2}$. The point P', which represents the complex number v, is the reflection of P in the ray $\arg(z) = \alpha$. You may assume that $\triangle OPQ \equiv \triangle OP'Q$.

(i) Write down the values of $|v|$ and $\arg(v)$. $\boxed{2}$

(ii) Hence show that $v = \overline{w}\,\mathrm{cis}\,2\alpha$. $\boxed{1}$

(iii) The circle $|z - (2 + 2i)| = 1$ is reflected in the ray $\arg(z) = \frac{\pi}{6}$. By using the result in part (ii), or otherwise, show that the equation of this new circle is $\boxed{2}$

$$\left|z - \left((1 + \sqrt{3}) + i(\sqrt{3} - 1)\right)\right| = 1.$$

Exam continues overleaf ...

2 a) i) $z + iw$

$= 3 - 4i + i(2+i)$

$= 3 - 4i + 2i - 1$

$= 2 - 2i$

ii) $z\bar{w}$

$= (3-4i)(2-i)$

$= 6 - 3i - 8i + 4i^2$

$= 2 - 11i$

b) (i) $|\alpha| = \sqrt{2}$

$\arg \alpha = -\frac{\pi}{4}$

$\therefore \alpha = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$

ii) $\alpha^4 + 4$

$= \sqrt{2}^4 \operatorname{cis}\left(4 \times -\frac{\pi}{4}\right) + 4$

$= (2^{\frac{1}{2}})^4 \operatorname{cis}(-\pi) + 4$

$= 2^2(\cos(-\pi) + i\sin(-\pi)) + 4$

$= 4(-1) + 4$

$= 0$

c) i) $w = 1 - \frac{2i}{x+iy} \times \frac{x-iy}{x-iy}$

$= 1 - \frac{2xi + 2i^2y}{x^2+y^2}$

$= \frac{x^2+y^2 - 2xi - 2y}{x^2+y^2}$

$= \left(\frac{x^2+y^2-2y}{x^2+y^2}\right) - i\left(\frac{2x}{x^2+y^2}\right)$

ii) $z = 0$

iii) ~~Im(w~~ $\operatorname{Re}(w) = 0$

$\therefore x^2+y^2-2y = 0$

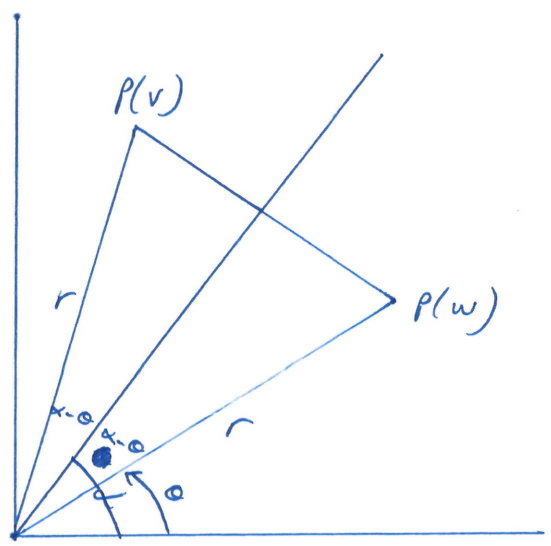
$x^2 + y^2 - 2y + 1 = 1$

$x^2 + (y-1)^2 = 1$

unit circle centre $(0,1)$

open circle at $(0,0)$

d)



i) $|v| = |w| = r$

$\arg(v) = \alpha + \alpha - \theta$

$= 2\alpha - \theta$

ii) $v = r[\operatorname{cis}(2\alpha - \theta)]$

$= r[\operatorname{cis}(2\alpha)\operatorname{cis}(-\theta)]$

$= r\operatorname{cis}(-\theta)\operatorname{cis}2\alpha$

$= \bar{w}\operatorname{cis}2\alpha$

$\bar{w} = r\operatorname{cis}(-\theta)$

de Moivre's Th.

$\operatorname{cis}(\theta_1)\operatorname{cis}(\theta_2) = \operatorname{cis}(\theta_1+\theta_2)$

$r_1\operatorname{cis}\theta_1 \, r_2\operatorname{cis}\theta_2 = r_1 r_2\operatorname{cis}(\theta_1+\theta_2)$

(iii) old centre = $2+2i$

$w = 2+2i$

$\alpha = \frac{\pi}{6}$

new centre (v)

$= \bar{w} \text{ cis } 2\alpha$

$= (2-2i) \text{ cis } \left(2 \times \frac{\pi}{6}\right)$

$= (2-2i)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= (2-2i)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= 1 + \sqrt{3}i - i + \sqrt{3}$

$= (1+\sqrt{3}) + i(\sqrt{3}-1)$

radius = 1 radius of circle is unchanged when reflected in the ray