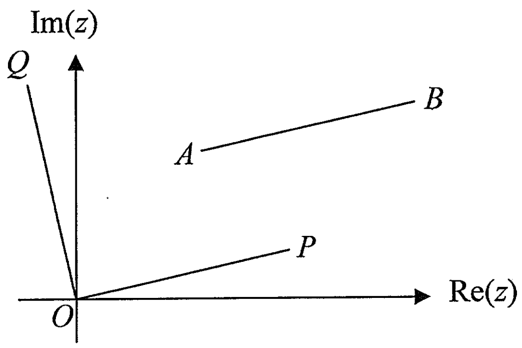
Question 2 (15 marks) **[START A NEW PAGE]**

(a) Find $\sqrt{21+20i}$ in the form $x + iy$. **3**

(b) Sketch the locus of z described by the inequality $|z-1+i| \le 1$ and state the minimum value of arg z. **2**

(c)



In the Argand diagram above, intervals AB, OP and OQ are equal in length, OP is parallel to AB and $\angle POQ = \frac{\pi}{2}$.

(i) If A and B represent the complex numbers $3 + 5i$ and $9 + 8i$ respectively, find the complex number which is represented by P. **1**

(ii) Hence find the complex number which is represented by Q. **1**

(d) If $z = x + iy \ (x, y \in R)$, find and describe in words, the locus of the points $P(x, y)$ such that $\operatorname{Im}\left(z + \frac{1}{z}\right) = 0$. **4**

(e) Write $\sqrt{3}+i$ and $\sqrt{3}-i$ in modulus/argument form. **4**
Hence show that $\left(\sqrt{3}+i\right)^{10} + \left(\sqrt{3}-i\right)^{10}$ is a rational number.

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Q 2

a) $(x+iy)^2 = 21+20i$

$x^2 + 2ixy - y^2 = 21 + 20i$

$x^2 - y^2 = 21 \qquad 2xy = 20$

$xy = 10$

$x^2 - \left(\frac{10}{x}\right)^2 = 21 \qquad y = \pm 2$

$x^2 - \frac{100}{x^2} = 21$

$x^4 - 21x^2 - 100 = 0$

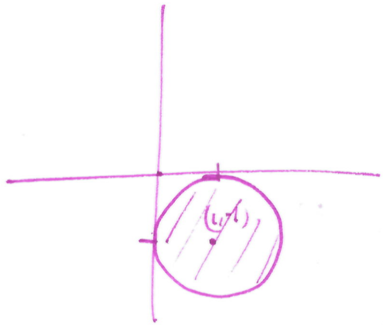
$(x^2-25)(x^2+4) = 0$

$x = \pm 5 \qquad x^2+4 \neq 0$ as $x \in R$

$\therefore \sqrt{21+20i}$

$= 5+2i, \quad -5-2i$

b) $|z-1+i| \leq 1 \Rightarrow |z-(1-i)| \leq 1$



$\min \arg z = -\frac{\pi}{2}$

(c) (i) $\overrightarrow{OP}$

$= \overrightarrow{AO} + \overrightarrow{OB}$

$= -(3+5i) + 9+8i$

$= 3i+6$ ✓

ii) $\overrightarrow{OQ}$

$= i(3i+6)$

$= -3+6i$ ✓

d) $z + \frac{1}{z}$

$= x+iy + \frac{1}{x+iy}$

$= x+iy + \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$= x+iy + \frac{x-iy}{x^2+y^2}$

$= \frac{(x^2+y^2)(x+iy)+x-iy}{x^2+y^2}$

$= \frac{x(x^2+y^2) + iy(x^2+y^2) + x - iy}{x^2+y^2}$

$= \frac{x(x^2+y^2+1) + iy(x^2+y^2-1)}{x^2+y^2}$

$Im\left(z+\frac{1}{z}\right)$

$= \frac{y(x^2+y^2-1)}{x^2+y^2}$

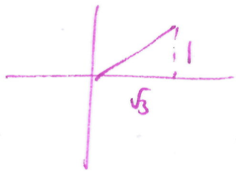
$= 0$

$\therefore y = 0 \quad$ or $\quad x^2+y^2-1=0$

$x^2+y^2 \neq 0 \qquad x^2+y^2=1$

$\therefore$ locus is a circle with the radius 1 and the x-axis except for the origin.

e) $z = \sqrt{3} + i$

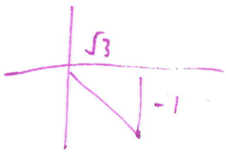


$|z| = 2$

$\arg z = \tan^{-1} \frac{1}{\sqrt{3}}$

$= \frac{\pi}{6}$

$w = \sqrt{3} - i$



$|w| = 2$

$\arg w = -\frac{\pi}{6}$

$\therefore z = 2 \operatorname{cis} \frac{\pi}{6}$

$\therefore w = 2 \operatorname{cis} \left(-\frac{\pi}{6}\right)$

$z^{10} = 2^{10} \operatorname{cis} \frac{10\pi}{6}$

$w^{10} = 2^{10} \operatorname{cis} \left(-\frac{10\pi}{6}\right)$

$$z^{10} + w^{10} = 2^{10} \left(\operatorname{cis} \frac{10\pi}{6} + \operatorname{cis}\left(-\frac{10\pi}{6}\right)\right)$$

$$= 2^{10} \left(\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} + \cos\left(-\frac{5\pi}{3}\right) + i \sin\left(-\frac{5\pi}{3}\right)\right)$$

$$= 2^{10} \left(2\cos \frac{5\pi}{3} + \cancel{i \sin \frac{5\pi}{3}} - \cancel{i \sin \frac{5\pi}{3}}\right)$$

$$= 2^{11} \times \frac{1}{2}$$

$$= 2^{10}$$