2005 CSSA

Question 2 **(Begin a new page)** **Marks**

(a) Find $\int \frac{1-\sin x}{\cos^2 x}\,dx$. 2

(b) Find $\int \left(e^x + e^{-\frac{1}{2}x}\right)^2\,dx$. 2

(c) Use the substitution $u = \sqrt{x}$ to evaluate $\int_1^{25} \frac{1}{x+\sqrt{x}}\,dx$, expressing the answer in simplest exact form. 3

(d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{1}{5-4\cos x}\,dx$, expressing the answer in simplest exact form. 3

(e)(i) If $I_n = \int_0^1 x(1-x)^n\,dx$, $n = 0, 1, 2, \ldots$, show that $I_n = \frac{n}{n+2} I_{n-1}$, $n = 1, 2, 3, \ldots$. 3

(ii) Hence show that $I_n = \frac{1}{2\,{}^{n+2}C_2}$, $n = 1, 2, 3, \ldots$ 2

Question 3 **(Begin a new page)**

(a) Show that the complex number $z = \frac{6-2i}{3+4i} - \frac{6}{5i}$ is real. 2

(b) $z_1 = 4\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$ and $z_2 = 2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$.

(i) On an Argand diagram draw the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ representing z_1, z_2, $z_1 + z_2$ respectively. 2

(ii) Hence find $|z_1 + z_2|$ in simplest exact form. 2

(c) The quadratic equation $z^2 + kz + 4 = 0$, k real and $-4 < k < 4$, has two non-real roots α, β.

(i) Explain why α, β are complex conjugates. Hence show that $|\alpha| = |\beta| = 2$. 2

(ii) If α, β have arguments $\frac{\pi}{4}$, $-\frac{\pi}{4}$, find the value of k. 2

(d)(i) On an Argand diagram shade the region where both $|z-(1+i)| \le \sqrt{2}$ and $0 \le \arg z \le \frac{\pi}{2}$ 2

(ii) Find the exact perimeter and the exact area of the shaded region. 3

2005 CSSA Q3

a) $$Z = \frac{6-2i}{3+4i} - \frac{6}{5i}$$

$$= \frac{6-2i}{3+4i} \times \frac{3-4i}{3-4i} - \frac{6}{5i} \times \frac{i}{i}$$

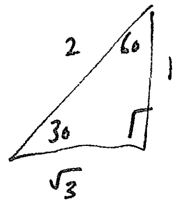
$$= \frac{18-24i-6i+8i^2}{9--16} - \frac{6i}{5i^2}$$

$$= \frac{10-30i}{25} - \frac{6i}{-5}$$

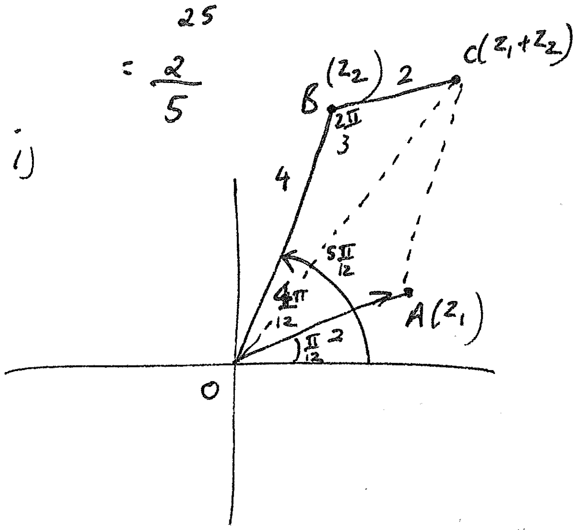
$$= \frac{10-30i}{25} + \frac{30i}{25}$$

$$= \frac{10}{25}$$

$$= \frac{2}{5}$$



b) i)



$$|z_1+z_2|^2$$

$$= 4^2 + 2^2 - 2 \times 4 \times 2 \times \cos\frac{2\pi}{3}$$

$$= 20 - 16\cos\frac{2\pi}{3}$$

$$= 20 - 16\left(-\cos\frac{\pi}{3}\right)$$

$$= 20 + 16\left(\frac{1}{2}\right)$$

$$= 28$$

$$|z_1+z_2| = 2\sqrt{7}$$

c) i) the coefficients of the equation are real

$\therefore$ complex roots occur in conjugate pairs

$\therefore$ α, β are complex conjugates

Let the roots be $\alpha, \bar{\alpha}$

$$\alpha + \bar{\alpha} = -k \qquad \alpha\bar{\alpha} = 4 \qquad |\alpha| = |\bar{\alpha}|$$

$$(x+iy)(x-iy) = 4$$

$$x^2 - i^2y^2 = 4$$

$$x^2 + y^2 = 4$$

$$\sqrt{x^2+y^2} = \sqrt{4}$$

$$|\alpha| = 2$$

$$\therefore |\bar{\alpha}| = 2$$

c) ii)

$\alpha + \bar{\alpha} = -k$

$x+iy + x-iy = -k$

$2x = -k$

$2\sqrt{2} = -k$

$k = -2\sqrt{2}$

$\alpha = 2 \operatorname{cis} \frac{\pi}{4}$

$\bar{\alpha} = 2 \operatorname{cis}\left(\frac{-\pi}{4}\right)$

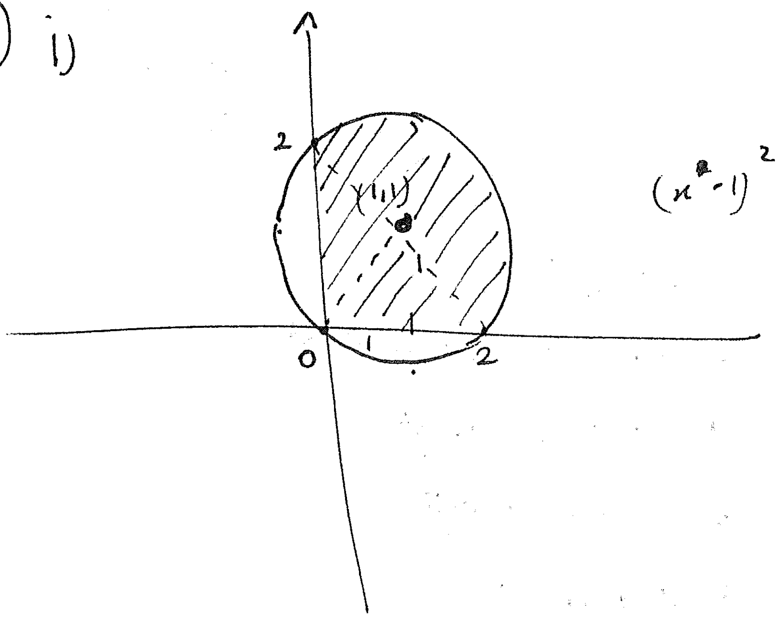
real $\alpha = 2\cos\frac{\pi}{4}$

$= \frac{2}{\sqrt{2}}$

$= \frac{2\sqrt{2}}{2}$

$= \sqrt{2}$

d) i)



$(x-1)^2 + (y-1)^2 = 2$

ii) $P = 2 + 2 + \frac{\pi D}{2}$

$= 4 + \frac{\pi \times 2\sqrt{2}}{2}$

$= 4 + \pi\sqrt{2}$

$A = \frac{1}{2} \times 2 \times 2 + \frac{\pi r^2}{2}$

$= 2 + \frac{\pi \times (\sqrt{2})^2}{2}$

$= 2 + \pi$